Permutation asymmetry inducing entanglement between degrees of freedom in multiphoton states

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We describe and examine entanglement between different degrees of freedom in multiphoton states based on the permutation properties. From the state description, the entanglement comes from the permutation asymmetry. According to the different permutation properties, the multiphoton states can be divided into several parts. It will help to deal with the multiphoton interference, which can be used as the measurement of the entanglement.

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I. INTRODUCTION

Photons interference has been widely applied in different protocols of quantum communication [1,2], quantum computation [3], and quantum metrology [4–7]. Many of those protocols utilize one degree of freedom (DOF), such as polarization, time (energy), momentum (path), etc., and neglect the relationship with others due to either filtering or the absence of correlations. Recently, however, more and more cases are discussed with multiple DOFs in the system, where a multidimensional correlated system can be remarkably formed [8]. In some cases, there is no entanglement between different DOFs, such as the hyperentanglement state [9–11]. In others, there may be entanglement [8]. In those states, entanglement will bring decoherence for one DOF when the other DOFs are discarded. It has been well described and observed for the two-photon correlated system [12].

Moreover, there are many cases focusing on the multiphoton system. The relationships between different DOFs and different photons are significantly more complicated than the two-photon cases. This is especially true for photons generated from parametric down conversion (PDC), where it is difficult to describe the state of more than one photon in the same mode of spatial DOF. There has been several experiments discussing the decoherence in the polarization DOF in the four-photon state from PDC [13,14]. The interpretation based on photons distinguishability has been well proposed [15,16]. Recently, a description based on photon permutation symmetry was proposed [17], where it described that entanglement between two DOFs decreased state purity and interference visibility. However, in that approach, it is not convenient to tell if there is entanglement between different DOFs by fully decomposing the state. Moreover, the description is appreciably more involved when there are multiphoton states in the same mode of spatial DOF.

In this paper, we will develop and introduce a convenient approach to determine the entanglement between different DOFs in a multiphoton state. The method is based on the permutation symmetry of different DOFs in the state, extending from permutation symmetry of different photons. Generally, the photons in one DOF are distinguishable because they can be recognized by the information of the other DOF. There are correlations between the two DOFs, which we call entanglement. On the other hand, the indistinguishable photons in one DOF should have a permutation symmetric form. Therefore, the entanglement is induced by the permutation asymmetry and can be read directly from the state which is described by photon creation operators. If there is no entanglement between different DOFs, the state can be described in a product form. Thus, a single DOF in different states will show the same interference behavior under the same operation if their descriptions are the same. For example, both the Greenberger-Horne-Zeilinger state (GHZ) [18] and the maximally entangled number state (NOON) [19,20] can be applied in the demonstration of the multiphoton de Broglie wavelength and the high resolution quantum phase measurement to approach the Heisenberg limit [19,21]. They will be described in Sec. II. When there is entanglement between different DOFs, the single DOF will not show perfect interference. In Sec. III, a four-photon state interference in the same spatial mode, which can be used as the measurement of the entanglement, will be described in detail. The four-photon state will be divided in several parts according to their permutation symmetries. It is much more convenient than other methods [17,21]. The last section is the conclusion.

II. MULTIPHOTON MULTI-DOF ENTANGLEMENT

To clearly describe the interference, the multiphoton state is written in a permutation symmetric form [17]

$$\prod_{i=1}^{N} a_i^{\dagger}|\mathrm{vac}\rangle = \frac{1}{\sqrt{N!}}\sum_P P(|a_1\rangle|a_2\rangle\cdots|a_N\rangle), \tag{1}$$

where P is the permutation operator that changes the positions of arbitrary two states. There are $N!$ terms for the N photons. For example, a two-photon state can be described as $a_1^{\dagger}a_2^{\dagger}|\mathrm{vac}\rangle=\Sigma_P P(|a_1\rangle|a_2\rangle)/\sqrt{2!}=(|a_1\rangle|a_2\rangle+|a_2\rangle|a_1\rangle)/\sqrt{2}$. For the identity case $|a_1\rangle=|a_2\rangle=|a\rangle$, $a^{\dagger 2}|\mathrm{vac}\rangle=\sqrt{2}|a\rangle|a\rangle$. If there is more than one uncoupled DOF in the system, the single-

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photon state is described as a product of single DOF,

$$a^{\dagger}(\alpha,\beta,\ldots,\gamma)|\text{vac}\rangle = |\alpha,\beta,\ldots,\gamma\rangle = |\alpha\rangle|\beta\rangle\cdots|\gamma\rangle, \quad (2)$$

where $\alpha,\beta,\ldots,\gamma$ represent the different DOFs.

Based on Eqs. (1) and (2), we can discuss the entanglement between different DOFs in a multiphoton state. The examples of two-photon and four-photon states with two DOFs have been presented in Ref. [17]. Generally, the N-photon state containing two DOFs can be written as [22]

$$|\Psi_N\rangle = \sum_{\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N} f(\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N)\prod_{k=1}^{N} a^{\dagger}(\alpha_k,\beta_k)|\text{vac}\rangle, \quad (3)$$

where α and β are the two DOFs.

In general, if there is no entanglement between the two DOFs, each DOF will have a permutation symmetric form. This is a result of permutation symmetry of bosonic particles. Thus, with this underlying principle, we can tell the entanglement based on the permutation symmetry of the state description.

Under the permutation of any photon's total wave function, the state described in Eq. (3) is invariant [23], such that

$$f(\ldots,\alpha_i,\beta_i,\ldots,\alpha_j,\beta_j,\ldots) = f(\ldots,\alpha_j,\beta_j,\ldots,\alpha_i,\beta_i,\ldots). \quad (4)$$

For a fixed set of α, if there is any permutation of the other DOF (β) satisfying

$$f(\ldots,\alpha_i,\beta_i,\ldots,\alpha_j,\beta_j,\ldots) = f(\ldots,\alpha_i,\beta_j,\ldots,\alpha_j,\beta_i,\ldots), \quad (5)$$

this part of the state can then be written as a product of permutation symmetric states,

$$\sum_{\beta_1,\ldots,\beta_N} f(\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N)\prod_{k=1}^{N} a^{\dagger}(\alpha_k,\beta_k)|\text{vac}\rangle \rightarrow \sum_{P}(|\alpha_1\rangle \times|\alpha_2\rangle,\ldots,|\alpha_N\rangle)\left(\sum_{\beta_1,\ldots,\beta_N} f(\alpha_1,\beta_1,\ldots,\alpha_N,\beta_N)\sum_{P} P(|\beta_1\rangle \times|\beta_2\rangle,\ldots,|\beta_N\rangle)\right). \quad (6)$$

If β DOF keeps the same description (described in the large parentheses in the above expression) for all of the permutation states of α DOF, the whole state can be written in a product form and there is no entanglement between the two DOFs. Otherwise, there is entanglement. Equation (4) describes the permutation symmetry of whole wave function for bosonic particles, while Eq. (5) describes the permutation symmetry of single DOF. It is a necessary condition for that there is no entanglement between different DOFs. The examples of two-photon states are discussed in detail in Ref. [17]. We note that the Bell singlet state is a special case in which both DOFs are in permutation antisymmetric form.

When there is no entanglement between different DOFs, a collective (on all qubits) operation on one DOF will have no effect on the other DOF and the photons in the DOF will show perfect interference. Moreover, if the description of one DOF is the same, the behavior under the same operation will be the same too.

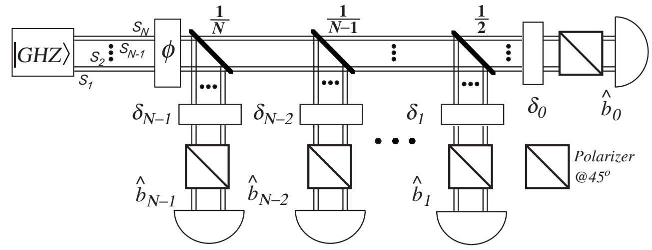

FIG. 1. Illustration of NOON state projection on a GHZ state. ϕ is the relative phase shift between two polarizations. Number above each beam splitter denotes the reflectivity. $\delta_k = 2k\pi/N$ is the phase delay between H and V polarizations. The polarizers are 45° oriented [20]. The detectors cover all N spatial modes.

For the multiphoton polarized state, if all photons are in the same mode of spatial DOF, there is no entanglement between the polarization DOF and the spatial DOF. The whole photon state can be written in a product form. For example, the NOON state is described as

$$\begin{aligned}|\text{NOON}\rangle &= (a_{H,S}^{\dagger N} + a_{V,S}^{\dagger N})|\text{vac}\rangle/\sqrt{2N!} \\ &= (|H\rangle^{\otimes N} + |V\rangle^{\otimes N}) \otimes |S\rangle^{\otimes N}/\sqrt{2}, \end{aligned} \quad (7)$$

where S is the spatial mode and H and V are horizontal and vertical polarizations, respectively. In addition, when N photons are different spatial modes, there also exists a product state that has no entanglement between two DOFs, such as the GHZ state,

$$\begin{aligned}|\text{GHZ}\rangle &= \left(\prod_{i=1}^{N} a_{H,S_i}^{\dagger} + \prod_{i=1}^{N} a_{V,S_i}^{\dagger}\right)|\text{vac}\rangle/\sqrt{2} \\ &= (|H\rangle^{\otimes N} + |V\rangle^{\otimes N}) \otimes \sum_{P} P(|S_1\rangle|S_2\rangle\cdots|S_N\rangle)/\sqrt{2N!}, \end{aligned} \quad (8)$$

where S_i are for the ith spatial modes. As shown in Eqs. (7) and (8), both the polarization DOF and the spatial DOF have the permutation symmetric form. Moreover, in the NOON state and N-photon GHZ state, the polarization DOF has the same form. If an operation acts collectively on this DOF, the two states will show the same results. For example, both of them will show the same application in the quantum phase measurement.

As we know, the NOON [19] state is a popular state for the quantum phase measurement. In the process, there is a relative phase shift ϕ between the two polarizations on all N photons which will cause a whole phase shift $e^{iN\phi}$. The result of NOON state projection [20,21,24] on the two states will show cosinusoidal oscillation for the N-photon de Broglie wavelength. In the NOON state projection measurement for the GHZ state, as shown in Fig. 1, each single-photon detector covers all N spatial modes. The N-fold coincidence counts will show the successful projection measurement, which can be described as

$$M = (|H\rangle^{\otimes N} - e^{-iN\phi}|V\rangle^{\otimes N})(\langle H|^{\otimes N} - e^{iN\phi}\langle V|^{\otimes N}) \otimes I_P, \quad (9)$$

where $I_P = \Sigma_P P(|S_1\rangle|S_2\rangle, \ldots, |S_N\rangle\langle S_1|\langle S_2|, \ldots, \langle S_N|)/N!$ is the matrix for the spatial DOF. Here we neglect the total coefficient from the photon loss of each beam splitter. The measurement result is

$$R = \langle \mathrm{GHZ}|M|\mathrm{GHZ}\rangle = (1 - \cos N\phi)/2. \quad (10)$$

Thus, the GHZ state can show the oscillation of de Broglie wavelength behavior. It can also be applied to the high resolution quantum phase measurement to approach the Heisenberg limit.

III. FOUR-PHOTON INTERFERENCE VISIBILITY WITH ENTANGLED DOFS

When there is entanglement between two DOFs, there will be distinguishability in one DOF, and will not be perfect interference in this DOF. The character of permutation symmetry can help to describe the distinguishability in the photon interference, even to calculate the interference visibility. Here we will utilize the different permutation properties to divide the state into several parts. The visibility calculation is much simplified with this method. As an example, we will discuss the four-photon interference in a single mode of spatial DOF.

As described in Refs. [20,21], the two-photon state from the two-cascaded type-I BBOs is expressed as

$$|\Psi_2\rangle = \frac{1}{\sqrt{2}}\sum_\alpha \varphi(\alpha)[a^{\dagger 2}(H,\alpha) + a^{\dagger 2}(V,\alpha)]|\mathrm{vac}\rangle$$
$$= \frac{1}{\sqrt{2}}(|HH\rangle + |VV\rangle)\left(\sum_\alpha \varphi(\alpha)|\alpha\alpha\rangle\right),$$

where α is for another DOF, such as frequency DOF. For simplicity, we assume $\varphi(\alpha)$ is real and $\Sigma_\alpha \varphi^2(\alpha) = 1$.

Correspondingly, the four-photon state is

$$|\Psi_4\rangle = \frac{1}{2}\left(\sum_\alpha \varphi(\alpha)[a^{\dagger 2}(H,\alpha) + a^{\dagger 2}(V,\alpha)]\right)^2|\mathrm{vac}\rangle. \quad (11)$$

However, there is a permutation asymmetric part in the above four-photon state, which induces the entanglement between the two DOFs. The state can be rewritten into two parts,

$$|\Psi_4\rangle = \tfrac{1}{2}(|\Psi_4\rangle_A + |\Psi_4\rangle_B), \quad (12)$$

where

$$|\Psi_4\rangle_A = \sum_\alpha \varphi(\alpha)[a^{\dagger 2}(H,\alpha)a^{\dagger 2}(H,\alpha) + a^{\dagger 2}(V,\alpha)a^{\dagger 2}(V,\alpha)$$
$$+ a^{\dagger 2}(H,\alpha)a^{\dagger 2}(V,\alpha) + a^{\dagger 2}(H,\alpha)a^{\dagger 2}(V,\alpha)]|\mathrm{vac}\rangle$$
$$+ \sum_{\alpha\neq\beta} \varphi(\alpha)\varphi(\beta)[a^{\dagger 2}(H,\alpha)a^{\dagger 2}(H,\beta)$$
$$+ a^{\dagger 2}(V,\alpha)a^{\dagger 2}(V,\beta)]|\mathrm{vac}\rangle$$
$$= (|HHHH\rangle + |VVVV\rangle)\left(\sum_\alpha \sqrt{24}\varphi^2(\alpha)|\alpha\alpha\alpha\alpha\rangle\right.$$

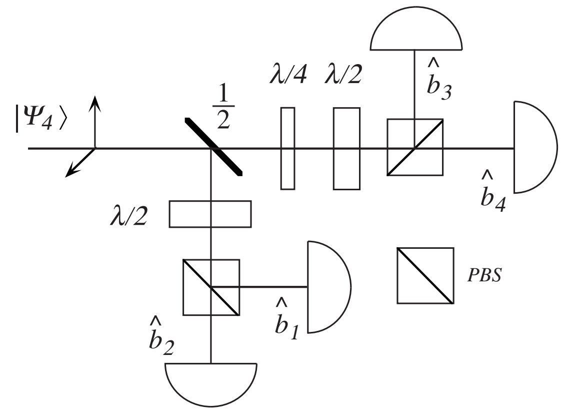

FIG. 2. NOON state projection measurement for the four-photon state. λ/2 and λ/4 are half-wave-plate at 22.5° and quarter-wave-plate at 0°.

$$\left. + \sum_{\alpha<\beta} \varphi(\alpha)\varphi(\beta)\sum_P P(|\alpha\alpha\beta\beta\rangle)/\sqrt{6}\right)$$
$$+ \sum_P P(|HHVV\rangle/\sqrt{6})\sum_\alpha \varphi^2(\alpha)|\alpha\alpha\alpha\alpha\rangle \quad (13)$$

is the permutation symmetric part, and

$$|\Psi_4\rangle_B = \sum_{\alpha\neq\beta} \varphi(\alpha)\varphi(\beta)[a^{\dagger 2}(H,\alpha)a^{\dagger 2}(V,\beta)$$
$$+ a^{\dagger 2}(H,\beta)a^{\dagger 2}(V,\alpha)]|\mathrm{vac}\rangle$$
$$= 2\sum_{\alpha<\beta} \varphi(\alpha)\varphi(\beta)[a^{\dagger 2}(H,\alpha)a^{\dagger 2}(V,\beta)$$
$$+ a^{\dagger 2}(H,\beta)a^{\dagger 2}(V,\alpha)]|\mathrm{vac}\rangle \quad (14)$$

is the permutation asymmetric part because of the absence of the photon state $a^\dagger(H,\alpha)a^\dagger(V,\alpha)a^\dagger(H,\beta)a^\dagger(V,\beta)|\mathrm{vac}\rangle$. In Eq. (13), the permutation symmetry state $\Sigma_P P(|iijj\rangle) = 4(|iijj\rangle + |ijij\rangle + |ijji\rangle + |jjii\rangle + |jiji\rangle + |jiij\rangle)$ is from the 24 permutation terms of $iijj$.

The mode in polarization DOF of each photon in $|\Psi_4\rangle_A$ is indistinguishable, while it is distinguishable in $|\Psi_4\rangle_B$. Thus, we can calculate the results of the two parts separately. For $|\Psi_4\rangle_A$, the polarized NOON state projection measurement [20], as shown in Fig. 2, has the form

$$M = (|HHHH\rangle - |VVVV\rangle)(\langle HHHH| - \langle VVVV|) \otimes I_{\alpha,\beta}, \quad (15)$$

where $I_{\alpha,\beta}$ is the identity matrix for the other DOF because there is no projection on this DOF in the measurement. This measurement is constructed based on Hanbury Brown–Twiss interferometer [25] by adding polarization projection before each detector. It is orthogonal with $\Sigma_P P(|HHVV\rangle/\sqrt{6})$ and will give null output, which is the result of Hong-Ou-Mandel interference for multiphoton [20]. After a phase shift ϕ, the measurements will show the perfect interference result, with oscillation for N-photon de Broglie wavelength for $|\Psi_4\rangle_A$, which are

$$R_A = {}_A\langle \Psi_4 | M | \Psi_4 \rangle_A$$
$$= 2(1-\cos 4\phi) \left\| \sum_\alpha \sqrt{24}\varphi^2(\alpha)|\alpha\alpha\alpha\alpha\rangle \right.$$
$$\left. + \sum_{\alpha<\beta} \varphi(\alpha)\varphi(\beta) \sum_P P(|\alpha\alpha\beta\beta\rangle)/\sqrt{6} \right\|^2$$
$$= (1-\cos 4\phi)(16+32K), \tag{16}$$

where $K=\Sigma_\alpha \varphi^4(\alpha) \leqslant 1$ indicates the entanglement of this DOF in the two-photon state [17,26].

The permutation asymmetric part $|\Psi_4\rangle_B$ cannot be written in the product state. There are six cases for the four-photon state to be detected by four detectors as shown in Fig. 2. However, the case that two H photons are detected by $\hat{b}_1\hat{b}_2$ and two V photons by $\hat{b}_3\hat{b}_4$ is canceled by the case of two H photons detected by $\hat{b}_3\hat{b}_4$ and two V photons by $\hat{b}_1\hat{b}_2$, since they have contrary phases. Therefore, the result of NOON state projection measurement is not difficult to calculate

$$R_C = \| |\Psi_4\rangle_B \|^2 \times 2/3 = 32(1-K) \times 2/3. \tag{17}$$

Thus, the total result is

$$R = R_A + R_B = 16(7+2K)\left(1 - \frac{3(1+2K)}{7+2K}\cos 4\phi\right)/3. \tag{18}$$

The interference visibility is $V=3(1+2K)/(7+2K)$. It is the same with the result of Eq. (75) in Ref. [20] if we set $K=\mathcal{E}/\mathcal{A}$. Therefore, the higher K, the higher interference visibility and the less entanglement between the two DOFs. When $K=1$, the visibility is 100%. In this case, there is no asymmetric part $|\Psi_4\rangle_B$ in the four-photon state and no entanglement between two DOFs. Therefore, this interferometric method can be used as the measurement of the entanglement.

IV. CONCLUSION

Based on permutation symmetry, we discussed the entanglement between different DOFs in a multiphoton state. Permutation asymmetry in the state description induces the entanglement between different DOFs. If a DOF does not entangle with other DOFs, the same state in this DOF will show the same interference behavior when an operation acts on all the photons collectively. As an example, we described that the GHZ state can also be used to approach Heisenberg limit in the quantum phase measurement. For the state which has entanglement between different DOFs, there is no maximal interference in one DOF. The visibility can be calculated by dividing the state into different parts according to their permutation properties. This method allows for the description of interference visibility significantly more conveniently in multiphoton multi-DOF states. Moreover, the interference can be used as the measurement of the entanglement.

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